Intersection Numbers on Moduli Spaces of Curves

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Joint Work with Professor Kefeng Liu

This talk is based on three joint papers with Prof. Kefeng Liu.

- **Intersection numbers and automorphisms of stable curves,** preprint, math.AG/0608209.
- **New properties of the intersection numbers on moduli spaces of curves,** preprint, math.AG/0609367.
- **The n-point functions for intersection numbers on moduli spaces of curves,** in preparation.

Main Topics of This Talk

We will present the following recently obtaind results.

- Our explicit formula of n-point functions for intersection numbers on moduli spaces of curves, generalizing Dijkgraaf's two-point function and Zagier's three-point function.
- Simplification of Faber's intersection number conjecture.
- New identities of intersection numbers that follows from n-point functions.
- Precise information of denominators of intersection numbers.
- Several conjectural numerical properties of intersection numbers.

Acknowledgements

My special thanks

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We would like to thank Professor Enrico Arbarello, Sergei Lando, Ravi Vakil, Edward Witten and Don Zagier for helpful comments and their interests in this work.

We thank Professor Carel Faber for his wonderful Maple programm for calculating Hodge integrals and for communicating Zagier's three-point function to us.

Moduli Spaces of Curves and Its Compactification

Fine moduli space of curves over $\mathbb{C}$ fail to exist due to automorphisms of curves (Riemann surfaces).

Constructions $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$

- The quotient of Teichmüller spaces by the action of the mapping class group.
- Geometric invariant theory (Gieseker, Mumford, etc).
- Deligne-Mumford-Knudsen compactification.

The coarse moduli space $\mathcal{M}_{g,n}$ is a dense open subset of its compactification $\overline{\mathcal{M}}_{g,n}$, which is projective. It's more natural and popular to treat $\overline{\mathcal{M}}_{g,n}$ as a Deligne-Mumford stack.

Intersection Theory on Moduli Spaces of Curves

Mumford (1974' Fields medalist) defined the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ on moduli spaces of curves in 1983.

It's more reasonable to consider the tautological subring $R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$. Since $A^*(\overline{\mathcal{M}}_{g,n})$ is too large and $R^*(\overline{\mathcal{M}}_{g,n})$ contains all geometrically natural classes.

Some tautological classes on $\overline{\mathcal{M}}_{g,n}$

- ψ_i the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the ith marked point.
- $\lambda_i = c_i(\mathbb{E})$ the ith Chern class of the Hodge bundle $\mathbb{E}$.
- κ classes originally defined by Miller-Morita-Mumford on $\overline{\mathcal{M}}_g$ and generalized to $\overline{\mathcal{M}}_{g,n}$ by Arbarello-Cornalba.

Netherlands School of Algebraic Geometry

There are many great Dutch mathematicians in history:

W. van Snell (1591-1626), C. Huygens (1629-1695), D.J. Korteweg (1848-1941), G. de Vries (1866-1934), L.E.J. Brouwer (1881-1966), B.L. van der Waerden (1903-1996), Edsger W. Dijkstra (1930-2002), etc.

In recent years, there are many world-renowned Dutch mathematicians working in algebraic geometry:

R. Dijkgraaf, C. Faber, G. van der Geer, A.J. de Jong, H. Lenstra, E. Looijenga, F. Oort, J. Steenbrink, A. Van de Ven, E. Verlinde, H. Verlinde, etc.

Witten's Conjecture and Kontsevich's Theorem

Correlation functions of 2D gravity

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \sum_{j=1}^{n} d_j = 3g - 3 + n.$$

The celebrated Witten's conjecture asserts that the generating function of these intersection numbers are governed by KdV hierarchy. Kontsevich (1998' Fields medalist) proved Witten's conjecture in 1992.

Witten's pioneering work revolutionized this field and motivated a surge of subsequent developments: Kontsevich's moduli spaces of stable maps, Faber's conjecture, Virasoro conjecture...

Reformulation of Witten's Conjecture

The following equivalent formulation of Witten's conjecture is due to Dijkgraaf-E.Verlinde-H.Verlinde.

DVV recursion formula

$$\langle\tilde\tau_k\prod_{j=1}^n\tilde\tau_{d_j}\rangle_g=\sum_{j=1}^n(2d_j+1)\langle\tilde\tau_{d_1}\dots\tilde\tau_{d_j+k-1}\dots\tilde\tau_{d_n}\rangle_g$$
$$+\frac{1}{2}\sum_{r+s=k-2}\langle\tilde\tau_r\tilde\tau_s\prod_{j=1}^n\tilde\tau_{d_j}\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{r+s=k-2\\ \underline{n}=I\coprod J}}\langle\tilde\tau_r\prod_{i\in I}\tilde\tau_{d_i}\rangle_{g'}\langle\tilde\tau_s\prod_{i\in J}\tilde\tau_{d_i}\rangle_{g-g'}$$

where $\langle\tilde\tau_{d_1}\dots\tilde\tau_{d_n}\rangle_g=\left[\prod_{i=1}^n(2d_i+1)!!\right]\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g$

Calculate Intersection Numbers

Witten-Kontsevich's recursion formula is the only feasible way known before to calculate all intersection numbers on moduli spaces of curves, which baffled mathematicians for many years.

Different Proofs of Witten's conjecture

- Kontsevich' combinatorial model for the intersection numbers and matrix integral.
- Via ELSV fomula that relates intersection numbers with Hurwitz numbers (Okounkov-Pandharipande, Kazarian-Lando, Chen-Li-Liu).
- Mirzakhani's recursion formula of Weil-Petersson volumes.
- Kim-Liu's proof by localization on moduli spaces of relative stable morphisms and an asymptotic analysis.

N-Point Functions of Intersection Numbers

Definition

We call the following generating function

$$F(x_1,\dots,x_n)=\sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{j=1}^{n}x_j^{d_j}$$

the n-point function.

Since n-point functions encoded all information of intersection numbers on moduli spaces of curves, a great deal of effort has been devoted to find explicit expressions of n-point function by many leading experts, including R. Dijkgraaf, D. Zagier, A. Okounkov.

Two and Three-Point Functions

We have one-point function $F(x) = \frac{1}{x^2}\exp(\frac{x^3}{24})$, R. Dijkgraaf's two-point function (about 1993)

$$F(x,y) = \frac{1}{x+y}\exp\left(\frac{x^3+y^3}{24}\right)\sum_{k=0}^{\infty}\frac{k!}{(2k+1)!}\left(\frac{1}{2}xy(x+y)\right)^k.$$

and Don Zagier's three-point function (before 1996)

$$F(x,y,z) = \exp\left(\frac{x^3+y^3+z^3}{24}\right)\sum_{r,s\geq 0}\frac{r!S_r(x,y,z)}{4^r(2r+1)!!}\frac{(\Delta/8)^s}{(r+s+1)!},$$

where $S_r(x,y,z)$ and Δ are the homogeneous symmetric polynomials defined by

$$S_r(x,y,z) = \frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{2(x+y+z)},$$

$$\Delta(x,y,z) = (x+y)(y+z)(z+x) = \frac{(x+y+z)^3}{3} - \frac{x^3+y^3+z^3}{3}.$$

Okounkov's Analytic Formula of N-Point Functions

In 2001, A. Okounkov (2006' Fields medalist) obtained an analytic expression of the n-point functions using n-dimensional error-function-type integrals. The key ingredient in Okounkov's formula is the following function

$$\mathcal{E}(x_1,\ldots,x_n) = \frac{1}{2^n \pi^{n/2}} \frac{\exp\left(\frac{1}{12}\sum x_i^3\right)}{\prod \sqrt{x_i}} \times \int_{s_i \geq 0} ds \exp\left(-\sum_{i=1}^{n} \frac{(s_i - s_{i+1})^2}{4x_i} - \sum_{i=1}^{n} \frac{s_i + s_{i+1}}{2} x_i\right)$$

Okounkov's proof borrows ingenious ideas from the theory of random matrices and random permutations.

Our Explicit Formula of N-Point Functions

Let $G(x_1,\ldots,x_n)=\exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right)\cdot F(x_1,\ldots,x_n)$ and $n\geq 2$.

$$G(x_1,\ldots,x_n)=\sum_{r,s\geq 0}\frac{(2r+n-3)!!P_r(x_1,\ldots,x_n)\Delta(x_1,\ldots,x_n)^s}{4^s(2r+2s+n-1)!!}$$

where P_r and Δ are homogeneous symmetric polynomials

$$\Delta=\frac{(\sum_{j=1}^n x_j)^3-\sum_{j=1}^n x_j^3}{3},$$

$$P_r=\left(\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2G(x_I)G(x_J)\right)_{3r+n-3}$$

$$=\frac{1}{2\sum_{j=1}^n x_j}\sum_{\underline{n}=I\coprod J}(\sum_{i\in I}x_i)^2(\sum_{i\in J}x_i)^2\sum_{r'=0}^{r}G_{r'}(x_I)G_{r-r'}(x_J).$$

Recovering Two and Three-Point Functions

Since $P_0(x,y)=\frac{1}{x+y}$, $P_r(x,y)=0$ for $r>0$ and

$$P_r(x,y,z)=\frac{r!}{2^r(2r+1)!}\cdot\frac{(xy)^r(x+y)^{r+1}+(\text{cyclic permutations})}{x+y+z},$$

we easily recover Dijkgraaf's two-point function and Zagier's three-point function obtained more than ten years ago.

Our n-point functions are derived from Witten's KdV equation. Although our derivation of n-point functions is very hard and painful, the verification is purely combinatorial and straightforward.

It is typical that from our n-point functions, many assertions about intersection numbers will be reduced to combinatorial identities.

Proof of Our N-Point Functions

We need to check that

$$E(x_1,\ldots,x_n) := (\sum_{j=1}^{n} x_j)\cdot G(x_1,\ldots,x_n)$$

satisfies the differential equation

$$2x_1\sum_{j=1}^{n} x_j\cdot\frac{\partial}{\partial x_1}E(x_1,\ldots,x_n)+\left(x_1+\frac{x_1^3}{4}\sum_{j=1}^{n}x_j+\sum_{j=1}^{n}x_j-\frac{x_1}{4}(\sum_{j=1}^{n}x_j)^3\right)E(x_1,\ldots,x_n)$$
$$=\frac{x_1}{2}\sum_{\underline{n}=I\coprod J}\left((\sum_{i\in J}x_i)^2+2(\sum_{i\in I}x_i)\cdot(\sum_{i\in J}x_i)\right)E(x_I)E(x_J).$$

and the initial value condition (i.e. the string equation)

$$G(x_1,\ldots,x_n,0) = (\sum_{j=1}^{n} x_j)\cdot G(x_1,\ldots,x_n).$$

New Way to Calculate Intersection Numbers

Our formula of n-point functions provides an elementary algorithm to calculate all intersection numbers on moduli spaces of curves other than the celebrated Witten-Kontsevich's recursion formula.

There is another slightly different formula of n-point functions whose coefficients look a little simpler (When $n = 3$, this has also been obtained by Zagier). For $n \geq 2$,

$$F(x_1, \ldots, x_n) = \exp \frac{(\sum_{j=1}^n x_j)^3}{24} \sum_{r,s \geq 0} \frac{P_r(x_1, \ldots, x_n) \Delta(x_1, \ldots, x_n)^s}{(-8)^s (2r + 2s + n - 1) s!}$$

Faber's Conjecture on Tautological Rings

In 1993, Carel Faber proposed his remarkable conjectures about the structure of tautological ring $\mathcal{R}^*(\mathcal{M}_g)$. Faber's conjecture motivated a tremendous progress toward understanding of the topology of moduli spaces of curves.

On Beijing ICM 2002, Faber's conjecture is mentioned as an important question in Kirwan's plenary lecture and the invited addresses of Pandharipande and Ionel.

Many leading experts have made important contributions to Faber's conjecture and its generalizations.

C. Faber, E. Getzler, I. Goulden, R. Hain, E. Ionel, D. Jackson, S. Morita, E. Looijenga, R. Pandharipande, R. Vakil, etc.

Faber's Intersection Number Conjecture

An important part of Faber's conjecture is the famous Faber's intersection number conjecture, which is the following relations in $\mathcal{R}^{g-2}(\mathcal{M}_g)$, if $\sum_{j=1}^{n} d_j = g - 2$,

$$\pi_*(\psi_1^{d_1+1} \dots \psi_n^{d_n+1}) = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)! \prod_{j=1}^{n}(2d_j+1)!!} \kappa_{g-2},$$

Faber's intersection number conjecture can be derived from the degree 0 Virasoro conjecture for $\mathbb{P}^2$, this is the work of Getzler and Pandharipande. In 2001, Givental announced a proof of Virasoro conjecture for $\mathbb{P}^n$. Y.P. Lee and R. Pandharipande are writing a book supplying the details.

Faber's Perfect Paring Conjecture

We hope that our n-point function will shed light on the Faber's perfect paring conjecture which is currently completely open. Faber has checked $g \leq 24$.

Faber's perfect paring conjecture

For $0 \leq i \leq g-2$, the natural product

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

is a perfect pairing.

Simplification of Faber's Intersection Number Conjecture

Theorem

Let $d_j \geq 1$ *and* $\sum_{j=1}^{n}(d_j - 1) = g - 2$*.*

$$\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\prod_{j=1}^{n}\tau_{d_j}\rangle_{g-1} = \frac{(2g-3+n)!}{2^{2g-1}(2g-1)!}\cdot\frac{2}{\prod_{j=1}^{n}(2d_j-1)!!}$$

Conjecture

Let $d_j \geq 0$*,* $\sum_{j=1}^{n} d_j = g + n - 2$ *and* $\underline{n} = \{1, 2, \ldots, n\}$*.*

$$\langle\tau_{d_1}\cdots\tau_{d_n}\tau_{2g}\rangle_g = \sum_{j=1}^{n}\langle\tau_{d_1}\cdots\tau_{d_{j-1}}\tau_{d_j+2g-1}\tau_{d_{j+1}}\cdots\tau_{d_n}\rangle_g$$

$$-\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}$$

Views From Leading Experts

I.P. Goulden, D.M. Jackson and R. Vakil, *The moduli space of curves, double Hurwitz numbers, and Faber's intersection number conjecture*, math.AG/0611659.

Quotation

However, it seems circuitous to prove the Intersection Number Conjecture by means of the Virasoro Conjecture. The latter is a very heavy instrument which conceals the combinatorial structure that lies behind the intersection numbers. As noted by K. Liu and Xu [LX], it is very desirable to have a shorter and direct explanation. (Liu and Xu show how the conjecture cleanly follows from another attractive conjectural identity.) For this reason, we give such an argument, paralleling our understanding of top intersection numbers on $\mathcal{M}_{g,n}$ via Hurwitz numbers.

New Identities of Intersection Numbers

Theorem

Let $K > 2g$, $d_j \geq 0$ and $\sum_{j=1}^n d_j = 3g + n - K - 1$.

$$\sum_{j=0}^{K} (-1)^j \langle \tau_{K-j} \tau_j \tau_{d_1} \cdots \tau_{d_n} \rangle_g = 0$$

Corollary

Let an even number $K > 2g$ and $\sum_{j=1}^n d_j = 3g + n - K - 2$.

$$\langle \tau_{d_1} \cdots \tau_{d_n} \tau_K \rangle_g = \sum_{j=1}^{n} \langle \tau_{d_1} \cdots \tau_{d_{j-1}} \tau_{d_j+K-1} \tau_{d_{j+1}} \cdots \tau_{d_n} \rangle_g$$

$$- \frac{1}{2} \sum_{\underline{n}=I \coprod J} \sum_{j=0}^{K-2} (-1)^j \langle \tau_j \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{K-2-j} \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}$$

Two Equivalent Conjectural Indentities I

Conjecture

Let $g \geq 2$, $d_j \geq 1$ *and* $\sum_{j=1}^{n}(d_j - 1) = g$.

$$\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!}\frac{1}{\prod_{j=1}^{n}(2d_j-1)!!} = \langle \tau_{d_1}\cdots\tau_{d_n}\tau_{2g-2}\rangle_g - \sum_{j=1}^{n}\langle \tau_{d_1}\cdots\tau_{d_{j-1}}\tau_{d_j+2g-3}\tau_{d_{j+1}}\cdots\tau_{d_n}\rangle_g + \frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-4}(-1)^j\langle \tau_j\prod_{i\in I}\tau_{d_i}\rangle\langle \tau_{2g-4-j}\prod_{i\in J}\tau_{d_i}\rangle$$

We have proved the above conjecture in the case of $n = 1$.

Two Equivalent Conjectural Indentities II

Since $(2g-3)! \cdot \mathrm{ch}_{2g-3}(\mathbb{E}) = (-1)^{g-1}(3\lambda_{g-3}\lambda_g - \lambda_{g-1}\lambda_{g-2})$, we have the following equivalent identity of Hodge integrals.

Conjecture

Let $g \geq 2$, $d_j \geq 1$ and $\sum_{j=1}^n (d_j - 1) = g$.

$$\frac{2g-2}{|B_{2g-2}|}\left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_{g-1}\lambda_{g-2} - 3\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}\lambda_{g-3}\lambda_g\right)$$
$$=\frac{1}{2}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_{2g-4-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1}$$
$$+\frac{(2g-3+n)!}{2^{2g+1}(2g-3)!}\cdot\frac{1}{\prod_{j=1}^n(2d_j-1)!!}$$

Two Quantities Related to Singularities

Definition

$$D_{g,n} = lcm\left\{ denom\left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right) \;\middle|\; \sum_{i=1}^{n} d_i = 3g-3+n \right\}$$

$$\mathcal{D}_g = lcm\left\{ denom\left(\int_{\overline{\mathcal{M}}_g} \kappa_{a_1} \cdots \kappa_{a_m} \right) \;\middle|\; \sum_{i=1}^{m} a_m = 3g-3 \right\}, g \geq 2$$

A neighborhood of $\Sigma \in \overline{\mathcal{M}}_{g,n}$ is of the form $U/\mathrm{Aut}(\Sigma)$, where U is an open subset of $\mathbb{C}^{3g-3+n}$. The denominators of intersection numbers all come from these orbifold quotient singularities, so the divisibility properties of $D_{g,n}$ and $\mathcal{D}_g$ should reflect overall behavior of singularities.

Motivations to Look at Denominators

Mumford, Towards an enumerative geometry of the moduli space of curves, 1983.

> *I believe that κ_l are the natural tautological classes to consider on $\overline{\mathcal{M}}_g$. On the other hand, the λ_l are natural classes for abelian varieties.*

Okounkov, Random trees and moduli of curves, 2003, LNM 1815.

> *. . . it is important to be aware of these automorphism issues (for example, to understand how intersection numbers on moduli spaces can be rational numbers) . . .*

Precise Information of Denominators

- $D_{g,n} \mid D_{g,n+1}$, $D_{g,n} \mid \mathcal{D}_g$. If $n \geq \lfloor \frac{g}{2} \rfloor + 1$, $D_{g,n} = \mathcal{D}_g$.
- $\mathrm{ord}(2, \mathcal{D}_g) = \mathrm{ord}(2, 24^g g!)$, $\mathrm{ord}(3, \mathcal{D}_g) = \mathrm{ord}(3, 24^g g!)$ and $\mathrm{ord}(p, \mathcal{D}_g) = \lfloor \frac{2g}{p-1} \rfloor$ for prime $p \geq 5$. (Exact values of $\mathcal{D}_g$)
- We order all Witten-Kontsevich τ-functions of given genus g

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prec \langle \tau_{k_1} \cdots \tau_{k_m} \rangle_g$$

  if $n < m$ or $n = m$ and there exists some i, such that $d_j = k_j$ for $j < i$ and $d_i < k_i$.

  If $5 \leq p \leq 2g + 1$ is a prime number, then the smallest tau function of genus g in the above lexicographical order that satisfies $\mathrm{ord}(p, denom\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g) = \lfloor \frac{2g}{p-1} \rfloor$ is

$$\langle \underbrace{\tau_{\frac{p-1}{2}} \cdots \tau_{\frac{p-1}{2}}}_{\lfloor \frac{2g}{p-1} \rfloor} \tau_d \rangle_g$$

Denominators of Intersection Numbers

Theorem

The denominator of intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

can only contain prime factors less than or equal to $2g+1$.

Theorem

If $p \geq 3$ *is a prime number, then*

$$\mathrm{ord}(p, denom\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g) \leq \mathrm{ord}(p, \prod_{i=1}^{n} (2d_i+1)!!).$$

Professor S. Lando's Inspiring Comments

We are grateful to Professor S. Lando for sending us the following very helpful comments.

The main question you address there, namely, what are the restrictions on the denominators of the intersection indices, seems for me to be a very important one. To my opinion, this question is of geometric nature: in which sense the orbifold moduli spaces of curves are, in fact, smooth spaces?

The existence of the restrictions on the denominators means that after being multiplied by a fixed integer (for a given genus), all of them become integral, whence reflecting the geometry of some "virtual" smooth spaces. It would be nice to understand further properties of these spaces.

The exact values of the constant $\mathcal{D}_g$ also are interesting and definitely reflect the underlying geometry.

Hints From Denominators of Intersection Numbers

The n-point function should contain $n-1$ parameters. Of course some parameters may be coupled!

In fact, our first belief that Dijkgraaf's two-point function and Zagier's three-point function can be generalized comes from the well behavior of denominators of intersection numbers.

Numerical Properties of Intersection Numbers

Conjecture

Let $h(\lambda)$ be a monomial of the form $\lambda_1^{k_1}\cdots\lambda_g^{k_g}$ and $d_1 < d_2$.

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\psi_2^{d_2}\cdots\psi_n^{d_n} h(\lambda) \leq \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1+1}\psi_2^{d_2-1}\cdots\psi_n^{d_n} h(\lambda)$$

Namely the more evenly $3g-3+n$ be distributed among indices, the larger the value of Hodge integrals. By string and dilaton equations, we may assume $d_3 \geq 2, \ldots, d_n \geq 2$.

In the most interesting case $h(\lambda)=1$, we have checked that the conjecture holds for $g \leq 20$. Moreover, for $n=2$, we have checked all $g \leq 1000$ (using Dijkgraaf's 2-point function); for $n=3$, we have checked all $g \leq 100$ (using Zagier's 3-point function).

More General Multinomial-Type Property

These seem to be very deep properties of intersection numbers.

Conjecture

Let $\langle \tau_{\underline{d}} \kappa_{\underline{a}} \lambda_{\underline{b}} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_{b_1} \cdots \lambda_{b_k}$.

- *If* $d_1 < d_2$, *then*
$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \kappa_{\underline{a}} \lambda_{\underline{b}} \rangle_{g,n} \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \kappa_{\underline{a}} \lambda_{\underline{b}} \rangle_{g,n}$$
- *If* $a_1 < a_2$, *then*
$$\langle \tau_{\underline{d}} \kappa_{a_1} \kappa_{a_2} \cdots \kappa_{a_m} \lambda_{\underline{b}} \rangle_{g,n} \leq \langle \tau_{\underline{d}} \kappa_{a_1+1} \kappa_{a_2-1} \cdots \kappa_{a_m} \lambda_{\underline{b}} \rangle_{g,n}$$
- *If* $b_1 < b_2$, *then*
$$\langle \tau_{\underline{d}} \kappa_{\underline{a}} \lambda_{b_1} \lambda_{b_2} \cdots \lambda_{b_k} \rangle_{g,n} \leq \langle \tau_{\underline{d}} \kappa_{\underline{a}} \lambda_{b_1+1} \lambda_{b_2-1} \cdots \lambda_{b_k} \rangle_{g,n}.$$

Interesting Implications

Intersection numbers of kappa classes are also called the higher Weil-Petersson volumes of the moduli space of curves.

Corollary

Let $a_j \geq 0$, $\sum_{j=1}^{m} a_j = 3g - 3 + n$ *and* $g \geq 1$.

$$\frac{(2g-2+n)^{m-1}}{24^g \cdot g!} \leq \langle \kappa_{a_1} \cdots \kappa_{a_m} \rangle_{g,n} \leq \frac{\langle \kappa_1^{3g-3+n} \rangle_{g,n}}{(2g-2+n)^{3g-3+n-m}}.$$

In particular, $\langle \kappa_1^{3g-3+n} \rangle_{g,n} \geq \frac{(2g-2+n)^{3g-4+n}}{24^g \cdot g!}$.

Corollary

Let $d_j \geq 0$ *and* $\sum_{j=1}^{n} d_j = 3g - 3 + n$.

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \geq \frac{1}{24^g \cdot g!}.$$

Thank You Very Much!